Annuities

Finite Math

10 January 2019

Compound Interest

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Example

The Russell Index tracks the average performance of various groups of stocks. On average, a $10,000 *investment in mid-cap growth funds over a* 10*-year period would have grown to* $63,000. *What annual nominal rate would produce the same growth if interest were compounded (a) annually, (b) continuously? Express answers as a percentage, rounded to three decimal places.*

Now You Try It!

Example

A promissory note will pay $\$50,000$ *at maturity* 6 *years from now. If you pay* $\$28,000$ *for the note now, what rate would you earn if interest were compounded (a) quarterly, (b) continuously?*

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Solution

(a) 9.78%

(b) 9.66%

Annuities

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At this point, we have only discussed investments where there was one initial deposit and a final payoff. But what if you make regular equal payments into an account? An *annuity* is a sequence of equal periodic payments. If payments are made at the end of each time interval, then the annuity if called an *ordinary annuity*. Our goal will be to find the future value of an annuity.

Future Value of an Annuity

Example

Suppose you decide to deposit $100 *every* 6 *months into a savings account which pays* 6% *compounded semiannually. If you make* 6 *deposits, one at the end of each interest payment period over the course of* 3 *years, how much money will be in the account after the last deposit is made?*

Solution

We can visualize the value of each of those \$100 deposits in a table.

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So adding up the future values of all these will give us the amount of money in the account

$$\begin{aligned} B &= \$100(1.03)^5 + \$100(1.03)^4 + \$100(1.03)^3 \\ &\quad + \$100(1.03)^2 + \$100(1.03) + \$100 = \$646.84 \end{aligned}$$

Future Value

Definition (Future Value of an Ordinary Annuity)

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Note that the payments are made at the end of each period.

Future Value

Example

What is the value of an annuity at the end of 10 *years if* $\$1,000$ *is deposited every* 3 *months into an account earning* 8% *compounded quarterly. How much of this value is interest?*

Now You Try It!

Example

If $\$1,000$ *is deposited at the end of each year for* 5 *years into an ordinary annuity earning* 8.32% *compounded annually, what will be the value of the annuity at the end of the* 5 *years?*

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Solution

$\$5,904.15$

Sinking Funds

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Definition (Sinking Funds)

$$PMT = FV \frac{r/m}{\left(1 + \frac{r}{m}\right)^n - 1}$$

where all the variables have the same meaning as for annuities.

Sinking Funds

Example

New parents are trying to save for their child's college and want to save up $80,000 *in* 17 *years. They have found an account that will pay* 8% *interest compounded quarterly. How much will they have to deposit every quarter in order to have a value of* $80,000*?*

Now You Try It!

Example

A bond issue is approved for building a marina in a city. The city is required to make regular payments every 3 *months into a sinking fund paying* 5.4% *compounded quarterly. At the end of* 10 *years, the bond obligation will be retired with a cost of* $\$5,000,000$. *How much will the city have to pay each quarter?*

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Solution

$\$95,094.67$

Present Value - Set Up

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Example

Example

How much should you deposit into an account paying 6% *compounded semiannually in order to be able to withdraw* \$2000 *every* 6 *months for* 2 *years? (At the end of the* 2 *years, there should be a balance of* \$0 *in the account.)*

Solution

This problem is solved similarly to how the future value of an annuity was, except this time, instead of finding the future value of each deposit, we have to find the present value of each withdraw.

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So adding up the present values of all these will give us the amount of money we should deposit into the account now

$$D=\$2000(1.03)^{-1}+\$2000(1.03)^{-2}+\$2000(1.03)^{-3}+\$2000(1.03)^{-4}=\$7434.20$$

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